Lyapunov Functions

Stilianos Louca

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Abstract

We introduce the notions of strong and weak, local and global Lyapunov functions for fixed points of real-time dynamical systems. LaSalle's Invariance Principle is proven and Lyapunov functions are presented for various special system classes. Finally, quadratic Lyapunov functions are presented as solutions to the continuous Lyapunov equation.

Contents

1 Preliminaries

We give here some basic definitions for dynamical systems, used in the rest of the article.

1.1 Definition: Dynamical system and fixed points

An abelian semi-group $(G, +)$ acting on a non-empty set $X \neq \emptyset$ is called a **semi-flow** on X. We call (X, G) a **dynamical system**. If X is a topological space, G a topological group and both the mapping induced by each $g \in G$ on X as well as the mapping $G \to X,\ g \mapsto g(x)$ induced by each start-point $x \in X$, are continuous, we call G a **continuous semi-flow** on X and (X, G) a **continuous dynamical system**. For any $x \in X$ we call the set $Gx := (g(x))_{g \in G}$ the **orbit** of x under the semi-flow G. A point $x_0 \in X$ is called a **fixed point** of the system if $g(x_0) = x_0$ for all $g \in G$. If

We call the system a **real-time** system if $G = [0, \infty)$ or $G = \mathbb{R}$. In that case we write $G_t : X \to X$ for the mapping induced on X by $t \in \mathbb{R}$. We call a set $A \subseteq X$ **positively invariant** to the semi-flow if $G_t(A) \subseteq A$ for all $t \geq 0$. For any point $x \in X$, we call $(G_t(x))_{t \geq 0}$ the **future orbit** of x under the semi-flow.

1.2 Definition: Fixed points

Let (X, G) be a real-time dynamical system. A point $x_0 \in X$ is called a **positively fixed point** if $G_t(x) = x$ for all $t \geq 0$. A positively fixed point x_0 is called **Lyapunov stable** if for any neighborhood U of x_0 there exists a neighborhood V of x_0, such that $x \in V$ implies $G_t(x) \in U\ \forall t \geq 0$. It is called **locally attracting** if there exists a neighborhood U of x_0 such that for all $x \in U$ one has $G_t(x) \overset{t \to \infty}{\longrightarrow} x_0$. It is called **globally attracting** if for all $x \in X$ one has $G_t(x) \overset{t \to \infty}{\longrightarrow} x_0$.

It is called **asymptotically stable** if it is both Lyapunov stable and locally attracting, otherwise it is called **unstable**. It is called **globally asymptotically stable** if it is Lyapunov stable and globally attracting.

If (X, d) is a metric space, then x_0 is called **globally exponentially stable** if there exists a constant $\lambda > 0$ and a continuous function $h : X \to [0, \infty)$ such that $d(G_t(x), x_0) \leq h(x) \cdot e^{-\lambda t}$ for all $x \in X$ and $t \geq 0$.

Remarks:

(i) Each globally attracting, positively fixed point is also locally attracting.

(ii) Each globally asymptotically stable, positively fixed point is also asymptotically stable.

(iii) Every globally exponentially stable, positively fixed point x_0 of the system is also globally asymptotically stable. Note that the continuity of h is generally needed to ensure the Lyapunov stability of x_0.

1.3 Definition: Limit points

Let (X, G) be a real-time dynamical system and $x_0 \in X$. We say $x \in X$ is a **future limit point** of x_0 if there exists a sequence $0 \leq t_1 < t_2 < .. \to \infty$ such that $G_{t_n}(x_0) \overset{n \to \infty}{\longrightarrow} x$. We call the set $G_\infty(x_0)$ of all future limit-points of x the **future limit set** of x_0.

1.4 Lemma: Positive invariance of future limit sets

Let (X, G) be a real-time, continuous dynamical system and $x_0 \in X$. Then the future limit set $G_\infty(x_0)$ is a positively invariant set.

Proof: If $G_{t_n}(x_0) \overset{n \to \infty}{\longrightarrow} x$ for some $x \in X$ and $0 \leq t_1 < t_2 < .. \to \infty$, then for every $t \geq 0$ one has

$$G_t(x) = G_t(\lim_{n \to \infty} G_{t_n}(x_0)) = \lim_{n \to \infty} G_t(G_{t_n}(x_0)) = \lim_{n \to \infty} G_{t_n + t}(x_0), \tag{1.1}$$

implying that $G_t(x)$ is a limit point of x_0 as well.

□

2 Lyapunov functions

We present Lyapunov functions for general real-time dynamical systems. Lyapunov functions are a special kind of auxiliary functions, that can be used to determine the stability behavior of positively fixed points of a system.

They generalize the notion of potential energy, which is minimized by the trajectory of a classical mechanical particle moving along its gradient. Once obtained, Lyapunov functions lead to a wealth of new insight into the local or even global behavior of a fixed point. Unfortunately, they are typically hard to find and often demand a great deal of *good luck*. See [2], [4] and [5] for more on Lyapunov functions.

2.1 Definition: Lyapunov Function

Let (X, G) be a real-time, continuous dynamical system and $x_0 \in X$ a positively fixed point. Let K be a sequentially compact neighborhood of x_0 and $H : K \to \mathbb{R}$ a continuous function. Then H is called a **weak, local Lyapunov-function** for x_0 if it satisfies:

- WL.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in K \setminus \{x_0\}$.
- WL.2 **Decreasing on orbits:** $H(G_t(x)) \leq H(x)$ for all $x \in X$ and $t \geq 0$ for which $x, G_t(x) \in K$.

It is called a **strong, local Lyapunov function** if it satisfies:

- SL.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in K \setminus \{x_0\}$.
- SL.2 **Strongly decreasing on orbits:** $H(G_t(x)) < H(x)$ for all $x \neq x_0$ and $t > 0$ for which $x, G_t(x) \in K$.

A continuous function $H : X \to \mathbb{R}$ is called a **weak, global Lyapunov function** if it satisfies:

- WG.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in X \setminus \{x_0\}$.
- WG.2 **Decreasing on orbits:** $H(G_t(x)) \leq H(x)$ for all $x \in X$ and $t \geq 0$.
- WG.3 **Radial non-boundedness:** For all non-relatively sequentially compact $A \subseteq X$ one has $\sup_{a \in A} |H(a)| = \infty$.

It is called a **strong, global Lyapunov function** if it is defined on X and satisfies:

- SG.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in X \setminus \{x_0\}$.
- SG.2 **Strongly decreasing on orbits:** $H(G_t(x)) < H(x)$ for all $x \neq x_0$ and $t > 0$.
- SG.3 **Radial non-boundedness:** For all non-relatively sequentially compact $A \subseteq X$ one has $\sup_{a \in A} |H(a)| = \infty$.

Remarks:

1. Every strong, local (global) Lyapunov function is also a weak, local (global) Lyapunov function.
2. Axiom (SG.3) is always satisfied if X is a sequentially compact space, since all of its subspaces are relatively sequentially compact.
3. If $X = \mathbb{R}^n$, then axiom (SG.3) is equivalent to $\lim_{\|x\| \to \infty} |H(x)| = \infty$.

2.2 Proposition: Lyapunov functions and positively invariant sets

Let (X, G) be a real-time, continuous dynamical system, $x_0 \in X$ a positively fixed point and H a Lyapunov function for x_0. Then:

1. If $H : X \to \mathbb{R}$ is weak, global Lyapunov, then for each $\varepsilon > 0$ the sets
$$X_\varepsilon := \{x \in X : H(x) \leq H(x_0) + \varepsilon\} \quad , \quad X_\varepsilon^o := \{x \in X : H(x) < H(x_0) + \varepsilon\} \tag{2.1}$$
are neighborhoods of x_0, positively invariant to the semi-flow G.
2. If $H : K \to \mathbb{R}$ is weak, local Lyapunov defined on the sequentially compact neighborhood K of x_0 and
$$\mu := \inf_{x \in \partial K} [H(x) - H(x_0)], \tag{2.2}$$
then $\mu > 0$ and for each $0 < \varepsilon < \mu$ the sets
$$K_\varepsilon := \{x \in K : H(x) \leq H(x_0) + \varepsilon\} \quad , \quad K_\varepsilon^o := \{x \in K : H(x) < H(x_0) + \varepsilon\} \tag{2.3}$$
are neighborhoods of x_0, positively invariant to the semi-flow G.

Proof:

1. Since X_ε^o is by continuity of H open, it is a neighborhood of x_0. Since it is contained in X_ε, the latter is one as well. Their positive invariance follows from axiom (WG.2).

2. By definition, the domain K is sequentially compact and the infimum μ is by continuity of H actually attained on ∂K. As $x_0 \notin \partial K$, by axiom (WL.1) $\mu > 0$. Now let $0 < \varepsilon < \mu$ and suppose that $G_{t_0}(x) \notin K_\varepsilon$ for some $x \in K_\varepsilon$ and $t_0 > 0$. By axiom (WL.2), H is decreasing on orbits so that this implies $G_{t_0}(x) \notin K$. As $t \mapsto G_t(x)$ is continuous, there exists a $0 < t \leq t_0$ such that $G_t(x) \in \partial K$. This means that $H(G_t(x)) - H(x_0) \geq \mu > \varepsilon$ and thus $G_t(x) \notin K_\varepsilon$, a contradiction to axiom (WL.2) and the fact that $G_t(x) \in K$.

   In the same manner one shows the positive invariance of K_ε^o.

   Since H is continuous, K_ε^o is open in K. As K is a neighborhood of x_0, K_ε^o is one as well. Since $K_\varepsilon^o \subseteq K_\varepsilon$, the set K_ε is also a neighborhood of x_0.

□

2.3 Lemma: Lyapunov functions and convergence of sequences

Let (X, G) be a real-time, continuous dynamical system. Let $x_0 \in X$ be a positively fixed point and H a weak or strong, local or global Lyapunov function for x_0. Then:

1. Every sequence $(x_n)_n \subseteq X$ within the domain of H, satisfying $H(x_n) \stackrel{n\to\infty}{\longrightarrow} H(x_0)$, converges towards x_0.

2. Suppose that the future orbit $(G_t(y_0))_{t\geq 0}$ of some $y_0 \in X$ is fully contained in the domain of H. Then H is constant on the future limit set $G_\infty(y_0)$ of y_0.

Proof:

1. By remark 2.1(1), it suffices to consider the case of H being weak, local or global Lyapunov. We begin with the case of H being a weak, global Lyapunov function. Suppose that $x_n \stackrel{n\to\infty}{\not\longrightarrow} x_0$, then there exists a neighborhood U of x_0 and a subsequence $(x_{n_k})_k \subseteq (x_n)_n$ such that $x_{n_k} \notin U\ \forall k$. As H is bounded on $(x_{n_k})_k$, by axiom (WG.3) the latter is relatively- sequentially compact and we can suppose it to be converging towards some point $y_0 \in X$. By continuity of H this implies $H(x_0) = \lim_{k\to\infty} H(x_{n_k}) = H(y_0)$. By axiom (WG.1) thus $y_0 = x_0$, a contradiction!

   Now let $H : K \to \mathbb{R}$ be a weak, local Lyapunov function and $H(x_n) \stackrel{n\to\infty}{\longrightarrow} H(x_0)$. As the sequence $(x_n)_n$ is already contained in the sequentially compact K, the above reasoning can still be applied.

2. As the function $[0, \infty) \to \mathbb{R}$, $t \mapsto H(G_t(y_0))$ is monotonically decreasing on the orbit of y_0 and bounded from below by $H(x_0)$, the limit $H_\infty := \lim_{t\to\infty} H(G_t(y_0))$ exists in $\mathbb{R}$. Thus, for any limit point $y = \lim_{n\to\infty} G_{t_n}(y_0)$, with $0 \leq t_1 < t_2 < .. \to \infty$, we find by continuity of H

$$H(y) = \lim_{n\to\infty} H(G_{t_n}(y_0)) = \lim_{t\to\infty} H(G_t(y_0)) = H_\infty. \tag{2.4}$$

□

2.4 Theorem: Lyapunov functions and Lyapunov stability

Let (X, G) be a continuous, real-time dynamical system. Let $x_0 \in X$ be a positively fixed point and H a weak or strong, local or global Lyapunov function for x_0. Then x_0 is Lyapunov stable.

Proof: By remark 2.1(1), it suffices to consider the case of H being weak, local or global Lyapunov.

- We begin with the case of H being a weak, global Lyapunov function. Let U be some arbitrary neighborhood of x_0. By proposition 2.2(1), each set of the form $X_\varepsilon^o := H^{-1}((-\infty, H(x_0) + \varepsilon))$ is a neighborhood of x_0, positively invariant to the semi-flow. It thus suffices to find an $\varepsilon > 0$ such that $X_\varepsilon^o \subseteq U$.

  Suppose such an $\varepsilon > 0$ does not exist. Then we can find a sequence $(x_n)_n \subseteq X \setminus U$ such that $H(x_n) \stackrel{n\to\infty}{\longrightarrow} H(x_0)$. By lemma 2.3(1) $x_n \stackrel{n\to\infty}{\longrightarrow} x_0$, a contradiction.

- Now consider the case of H being a weak, local Lyapunov function defined on the sequentially compact neighborhood K of x_0. Define $\mu > 0$ as in (2.2). Then by proposition 2.2(2), the set

$$K_\varepsilon^o := \left\{x \in K : H(x) \leq H(x_0) + \varepsilon\right\} \tag{2.5}$$

is for every $0 < \varepsilon < \mu$ a neighborhood of x_0, positively invariant to the semi-flow. It thus suffices to find an $\varepsilon > 0$ such that $K_\varepsilon^o \subseteq U$. Suppose such an $\varepsilon > 0$ does not exist. Then we can find a sequence $(x_n)_n \subseteq K \setminus U$ such that $H(x_n) \overset{n\to\infty}{\longrightarrow} H(x_0)$. By lemma 2.3(1) $x_n \overset{n\to\infty}{\longrightarrow} x_0$, a contradiction.

□

2.5 Theorem: Strong, local Lyapunov functions and asymptotic stability

Let (X, G) be a continuous, real-time dynamical system. Let $x_0 \in X$ be a positively fixed point and $H : K \to \mathbb{R}$ a strong, local Lyapunov function for x_0. Then x_0 is asymptotically stable.

Proof: By theorem 2.4 x_0 is Lyapunov stable. What is left to show, its its local attractiveness.

- Let $\mu > 0$ defined as in Eq. (2.2) and $0 < \varepsilon < \mu$ chosen arbitrarily. Define $K_\varepsilon \subseteq K$ as in Eq. (2.3), then by proposition 2.2(2) it is a sequentially compact neighborhood of x_0, positively invariant to the semi-flow. Now let $x \in K_\varepsilon$, then by axiom (SL.2) the function $[0,\infty) \to \mathbb{R},\ t \mapsto H(G_t(x))$ is monotonically decreasing and is by axiom (SL.1) bounded from below by $H(x_0)$. Thus $H_\infty := \lim_{t\to\infty} H(G_t(x))$ exists in $\mathbb{R}$.

- We show that $G_t(x) \overset{t\to\infty}{\longrightarrow} x_0$. By lemma 2.3(1), it suffices to show that $H_\infty = H(x_0)$.

  Indeed, as the orbit $(G_t(x))_{t\geq 0}$ is contained in the sequentially compact set K_ε, there exist $t_1 < t_2 < .. \to \infty$ and some point $y_0 \in K_\varepsilon$ such that $G_{t_n}(x) \overset{n\to\infty}{\longrightarrow} y_0$. By continuity of H this implies $H_\infty = \lim_{n\to\infty} H(G_{t_n}(x)) = H(y_0)$. On the other hand one has

$$\begin{aligned} H(G_1(y_0)) &= H(G_1 \lim_{n\to\infty} G_{t_n}(x)) \overset{(\clubsuit)}{=} \lim_{n\to\infty} H(G_1(G_{t_n}(x))) \\ &= \lim_{n\to\infty} H(G_{t_n+1}(x)) = \lim_{t\to\infty} H(G_t(x)) \\ &= H(y_0), \end{aligned} \tag{2.6}$$

  whereas in ($\clubsuit$) we used the continuity of H and G_1. By axiom (SG.2) this implies $y_0 = x_0$ and thus $H_\infty = H(x_0)$.

□

2.6 Theorem: Strong, global Lyapunov functions and global, asymptotic stability

Let (X, G) be a continuous, real-time dynamical system. Let $x_0 \in X$ be a positively fixed point and $H : X \to \mathbb{R}$ a strong, global Lyapunov function for x_0. Then x_0 is the unique positively fixed point of the system and globally, asymptotically stable.

Proof: By theorem 2.4, x_0 is Lyapunov stable. We thus show its uniqueness and global attractiveness.

- We begin by showing the uniqueness of x_0 as a fixed point. Let $x \in X$ be another fixed point of the system, then $H(G_t(x)) = H(x)$ for all $t > 0$, which by axiom (SG.2) implies $x = x_0$.

- Now let $(x(t))_{t\geq 0}$ be some arbitrary orbit of the system. We show that the limit $H_\infty := \lim_{t\to\infty} H(x(t))$ exists in $\mathbb{R}$.

  Indeed, the function $[0,\infty) \to \mathbb{R},\ t \mapsto H(x(t))$ is by axiom (SG.2) monotonically decreasing and by axiom (SG.1) bounded below by $H(x_0)$, so that $H_\infty := \lim_{t\to\infty} H(x(t))$ exists with $H_\infty \geq H(x_0)$.

- We show that $x(t) \overset{t\to\infty}{\longrightarrow} x_0$. By lemma 2.3(1), it suffices to show that $H_\infty = H(x_0)$.

  As H is bounded on the orbit $(x(t))_{t\geq 0}$, by axiom (SG.3) the orbit is relatively sequentially compact, that is, contained in a sequentially compact set. Therefore, there exist $t_1 < t_2 < .. \to \infty$ and some point

$y_0 \in X$ such that $x(t_n) \overset{n\to\infty}{\longrightarrow} y_0$. By continuity of H this implies $H_\infty = \lim_{n\to\infty} H(x(t_n)) = H(y_0)$. On the other hand one has

$$\begin{aligned} H(G_1(y_0)) &= H(G_1 \lim_{n\to\infty} x(t_n)) \overset{(\clubsuit)}{=} \lim_{n\to\infty} H(G_1(x(t_n))) \\ &= \lim_{n\to\infty} H(x(t_n+1)) = \lim_{t\to\infty} H(x(t)) \\ &= H(y_0), \end{aligned} \tag{2.7}$$

whereas in ($\clubsuit$) we used the continuity of H and G_1. By axiom (SG.2) this implies $y_0 = x_0$ and thus $H_\infty = H(x_0)$.

□

3 Lyapunov functions for differentiable systems

3.1 Definition: Differentiable dynamical systems

We now consider a real-time dynamical system (X, G) on an n-dimensional differentiable manifold X, with G induced by the continuous vector field $f : X \to TX$, that is as a solution of $\frac{d}{dt} G_t(x) = f(G_t(x))$ with $G_0(x) = x$. We call (X, G) a **differentiable dynamical system** if each $G_t : X \to X$ is well-defined and continuous for all times $t \geq 0$.

Remarks:

(i) Every differentiable dynamical system is also a continuous one.

3.2 Proposition: Global Lyapunov functions for differentiable systems

Let (X, G) be a differentiable dynamical system induced by the continuous vector field f on the differentiable manifold X. Let $x_0 \in X$ be some point and $H : X \to \mathbb{R}$ a differentiable function satisfying:

DWG.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in X \setminus \{x_0\}$.

DWG.2 **Weakly decreasing on orbits:** $f_x H \leq 0$ for all $x \in X$.

DWG.3 **Radial non-boundedness:** For all non-relatively compact $A \subseteq X$ one has $\sup_{a\in A} |H(a)| = \infty$.

Then x_0 is a fixed point of the system and H a weak, global Lyapunov function for x_0. Alternatively, suppose that H satisfies:

DSG.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in X \setminus \{x_0\}$.

DSG.2 **Strongly decreasing on orbits:** $f_x H < 0$ for all $x \in X \setminus \{x_0\}$.

DSG.3 **Radial non-boundedness:** For all non-relatively compact $A \subseteq X$ one has $\sup_{a\in A} |H(a)| = \infty$.

Then H is a strong, global Lyapunov function for the fixed point x_0.

Proof: As x_0 is by axiom (DWG.1) a local extremum of the differentiable H, all of its directional derivatives vanish at x_0. Thus $f_{x_0} H = 0$ and axiom (DSG.2) implies axiom (DWG.2). In any case, axiom (DWG.2) implies that $H(G_t(x)) \leq H(x)$ for all $x \in X$ and $t \geq 0$. Therefore, x_0 is by axiom (DWG.1) necessarily a fixed point and H a weak, global Lyapunov function for x_0.
In case of axiom (DSG.2), the derivative of H along the vector field f is negative everywhere but x_0, so that for each $x \in X \setminus \{x_0\}$ there exists an $\varepsilon > 0$ with $H(G_\varepsilon(x)) < H(x)$. By the above considerations this actually implies that $H(G_t(x)) < H(x)$ for all $t > 0$, so that H is a strong Lyapunov function.

□

Remarks:

(i) Suppose that $H : K \to \mathbb{R}$ is a weak or strong, global Lyapunov function for the positively fixed point $x_0 \in X$. Then if H is continuously differentiable, it satisfies axioms (DWG.1), (DWG.2) and (DWG.3).

3.3 Proposition: Local Lyapunov functions for differentiable systems

Let (X, G) be a differentiable dynamical system induced by the continuous vector field f on the differentiable manifold X. Let $x_0 \in X$ be some point and K a sequentially compact neighborhood of x_0. Let $H : K \to \mathbb{R}$ be a continuous function, differentiable on the interior K^o, satisfying:

DWL.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in K \setminus \{x_0\}$.

DWL.2 **Weakly decreasing on orbits:** $f_x H \leq 0$ for all $x \in K^o$.

Then x_0 is a positively fixed point of the system. Furthermore, there exists a sequentially compact neighborhood $\widetilde{K}$ of x_0 (possibly smaller than K), such that $H\big|_{\widetilde{K}}$ is a weak, local Lyapunov function for x_0. Alternatively, suppose that H satisfies:

DSL.1 **Positive definiteness:** $H(x_0) < H(x)$ for all $x \in K \setminus \{x_0\}$.

DSL.2 **Strongly decreasing on orbits:** $f_x H < 0$ for all $x \in K^o \setminus \{x_0\}$.

Then x_0 is a positively fixed point of the system. Furthermore, there exists a sequentially compact neighborhood $\widetilde{K}$ of x_0 (possibly smaller than K), such that $H\big|_{\widetilde{K}}$ is a strong, local Lyapunov function for x_0.

Proof: As $x_0 \in K^o$ is by axiom (DWL.1) a local extremum of the differentiable H, all of its directional derivatives vanish at x_0. Thus $f_{x_0} H = 0$ and axiom (DSL.2) implies axiom (DWL.2). In any case, axiom (DWL.2) implies that $H(G_t(x)) \leq H(x)$ for all $x \in K^o$ and $t \geq 0$ for which $G_{[0,t]}(x) \subseteq K^o$.
Now if x_0 is not a positively fixed point, there must by continuity of the dynamical system exist an $\varepsilon > 0$ such that $G_\varepsilon(x_0) \neq x_0$ and $G_{[0,\varepsilon]}(x_0) \subseteq K^o$. Thus $H(G_\varepsilon(x_0)) \leq H(x_0)$. But by axiom (DWL.1) this is a contradiction. Therefore, x_0 is a positively fixed point.
As K is sequentially compact, the infimum

$$\mu := \inf_{x \in \partial K} [H(x) - H(x_0)] \tag{3.1}$$

is attained. As $x_0 \notin \partial K$, one has by axiom (DWL.1) $\mu > 0$. Choose any $0 < \varepsilon < \mu$ and consider the set

$$K_\varepsilon := \{x \in K : H(x) \leq H(x_0) + \varepsilon\}. \tag{3.2}$$

Then, since H is continuous, K_ε is a sequentially compact neighborhood of x_0. Note that $K_\varepsilon \subseteq K^o$ since $K_\varepsilon \cap \partial K = \emptyset$ (by definition of μ). Furthermore, K_ε is positively invariant to the semi-flow. Indeed, if $G_t(x) \notin K_\varepsilon$ for some $x \in K_\varepsilon$ and $t > 0$, then there exists by continuity of the system and H a $t_0 \geq 0$ such that $G_{t_0}(x) \in K_{\widetilde{\varepsilon}} \setminus K_\varepsilon$ and $G_{[0,t_0]}(x) \subseteq K_{\widetilde{\varepsilon}} \subseteq K^o$ for some $\widetilde{\varepsilon} \in (\varepsilon, \mu)$. Otherwise said, $H(G_{t_0}(x)) > H(x)$, even though $G_{[0,t_0]}(x) \subseteq K^o$, a contradiction!
We consider the restriction of H to K_ε. Then $H\big|_{K_\varepsilon}$ is weak, locally Lyapunov for x_0 if axiom (DWL.2) is satisfied. Indeed, for any $x \in K_\varepsilon$ one has $G_t(x) \in K_\varepsilon$ for all $t \geq 0$ and thus by the above $H(G_t(x)) \leq H(x)$ for all $t \geq 0$.
On the other hand, it is strong, locally Lyapunov for x_0 if axiom (DSL.2) is satisfied. Indeed, for any $x \in K_\varepsilon \setminus \{x_0\}$ one has $G_t(x) \in K_\varepsilon$, so that the function $[0, \infty) \to \mathbb{R},\ t \mapsto H(G_t(x))$ is monotonically decreasing. But since at least locally around x, H is strictly decreasing along the orbit, one has $H(G_t(x)) < H(x)$ for all $t > 0$.

□

Remarks:

(i) Suppose that $H : K \to \mathbb{R}$ is a weak or strong, local Lyapunov function for the positively fixed point $x_0 \in X$. Then if H is continuously differentiable, it satisfies axioms (DWL.1) and (DWL.2).

3.4 Theorem: LaSalle's Invariance Principle

Let (X, G) be a differentiable dynamical system induced by the continuous vector field f on the differentiable manifold X. Let x_0 be a positively fixed point and $H : D(H) \to \mathbb{R}$ a weak or strong, local or global Lyapunov function for x_0. Let $\mathcal{I}$ be the union of all positively invariant sets contained in $\mathcal{S} := \{x \in D(H) : f_x H = 0\}$ (or equivalently, the union of all future orbits fully contained in $\mathcal{S}$). Let $U \subseteq D(H)$ be some positively invariant set. Then the future limit set $G_\infty(y)$ of every $y \in U$ is contained in $\mathcal{I}$.

Proof: Let $y \in U$. By lemma 1.4 $G_\infty(y)$ is a positively invariant set. By lemma 2.3(2), H is constant on $G_\infty(y)$, so that $f_x H = 0$ for every $x \in G_\infty(y)$. This finishes the proof.

□

Examples:

(i) Suppose H to be a global Lyapunov function, that is $D(H) = X$. Then the future limit set $G_\infty(y)$ of every $y \in X$ is contained in $\mathcal{I}$.

(ii) Suppose H to be a local Lyapunov function, that is with $D(H) =: K$ sequentially compact. Let μ be defined as in Eq. (2.2) and K_ε defined as in Eq. (2.3) for some $0 < \varepsilon < \mu$. Then $K_\varepsilon \subseteq K$ is by proposition 2.2(2) positively invariant, so that by LaSalle's principle each $y \in K_\varepsilon$ has its future limit set $G_\infty(y)$ contained in $\mathcal{I}$.

3.5 Corollary of LaSalle's Invariance Principle

Let (X, G) be a differentiable dynamical system induced by the continuous vector field f on the differentiable manifold X. Let x_0 be a positively fixed point and $H : D(H) \to \mathbb{R}$ a weak or strong, local (global) Lyapunov function for x_0. Let $\mathcal{I}$ be the union of all positively invariant sets contained in $\mathcal{S} := \{x \in D(H) : f_x H = 0\}$. If $\mathcal{I}$ only contains x_0, then x_0 is (globally) asymptotically stable.

Proof:

- We begin with the case of H being a global Lyapunov function. Then by LaSalle, more precisely example 3.4(i), the future limit set of each $y \in X$ can only contain x_0. Now as H is monotonically decreasing on the future orbit of y and bounded below by $H(\omega_0)$, it is bounded on $(G_t(y))_{t\geq 0}$. Since it is radially bounded, $(G_t(y))_{y\geq 0}$ is contained in a sequentially compact set. Now suppose $G_t(y) \overset{t\to\infty}{\not\longrightarrow} x_0$, then we can find $t_1 < t_2 < .. \to \infty$ such that the sequence $(G_{t_n}(y))_n$ converges to some point other than x_0. But this is a contradiction to $G_\infty(y) \subseteq \{x_0\}$.
- Now consider the case of H being locally Lyapunov, thus only defined on some sequentially compact $D(H) = K$. By proposition 2.2(2), there exists some sequentially compact, positively invariant neighborhood $K_\varepsilon \subseteq K$ of x_0. By LaSalle 3.4, the future limit set of each $y \in K_\varepsilon$ can only contain x_0. Now suppose $G_t(y) \overset{t\to\infty}{\not\longrightarrow} x_0$, then we can find $t_1 < t_2 < .. \to \infty$ such that the sequence $(G_{t_n}(y))_n$ converges to some point other than x_0. But this is a contradiction to $G_\infty(y) \subseteq \{x_0\}$.

□

3.6 Example: The damped harmonic oscillator

In the following we study the stability of the unique, positively fixed point of the harmonic and damped harmonic oscillator. We identify the first and second variable with the *position* and *velocity* of the oscillator respectively.

1. We consider the differentiable dynamical system on $X := \mathbb{R}^2$ induced by the vector field $f_\mathbf{x} = (x_2, -\omega^2 x_1)^T$, corresponding to an undamped harmonic oscillator of frequency $\omega > 0$. The origin $\mathbf{x}_0 := 0$ is the unique positively fixed point of the dynamical system. Now consider the function $H : X \to \mathbb{R}$ defined as $H(\mathbf{x}) := \omega^2 x_1^2 + x_2^2$. Then H is radially unbounded, that is satisfies axiom (DWG.1). Furthermore $H(\mathbf{x}) > H(\mathbf{x}_0)$ for every $\mathbf{x} \neq \mathbf{x}_0$, that is H satisfies axiom (DWG.2). Finally

$$f_\mathbf{x} H = \nabla_\mathbf{x} H \cdot f_\mathbf{x} = \left(2\omega^2 x_1,\ 2x_2\right) \cdot \begin{pmatrix} x_2 \\ -\omega^2 x_1 \end{pmatrix} = 0, \tag{3.3}$$

   for all $\mathbf{x} \in X$, that is H satisfies axiom (DWG.3) and is thus by 3.2 a weak, global Lyapunov function for $\mathbf{x}_0$. By theorem 2.4 $\mathbf{x}_0$ is Lyapunov stable. By proposition 2.2(1) each ellipsoid $K_\varepsilon := \left\{\mathbf{x} \in \mathbb{R}^2 : H(\mathbf{x}) \leq \varepsilon\right\}$, with $\varepsilon > 0$, is a positively invariant set. Nonetheless, $\mathbf{x}_0$ is **not** locally attracting, as for any arbitrarily small $\varepsilon > 0$, the trajectory starting at $(\varepsilon, 0)$ stays on the contour $H(\mathbf{x}) = \omega^2\varepsilon^2$, of which $\mathbf{x}_0 = 0$ is not a closure point.

2. Consider the *damped* version $f_\mathbf{x} = (x_2, -\alpha x_2 - \omega^2 x_1)$, where $\alpha > 0$ is some constant corresponding to a velocity-proportional *friction*. Then the original Lyapunov function satisfies

$$f_\mathbf{x} H = \left(2\omega^2 x_1,\ 2x_2\right) \cdot \begin{pmatrix} x_2 \\ -\alpha x_2 - \omega^2 x_1 \end{pmatrix} = -2\alpha x_2^2 \leq 0, \tag{3.4}$$

and is thus *only* a weak, global Lyapunov function for $\mathbf{x}_0$. Nonetheless, the only positively invariant set completely contained in $\mathcal{S} := \{\mathbf{x} \in X : f_{\mathbf{x}}H = 0\} = \{(0, x_2) : x_2 \in \mathbb{R}\}$ is $\{\mathbf{x}_0\}$ its self. By LaSalle's Invariance Principle 3.5, $\mathbf{x}_0$ is globally asymptotically stable.

3. Now consider the non-linearly damped version $f_{\mathbf{x}} = (x_2, -\alpha x_2^3 - \omega^2 x_1)$, where $\alpha > 0$ is some constant. Then the original Lyapunov function satisfies

$$f_{\mathbf{x}}H = \left(2\omega^2 x_1,\ 2x_2\right) \cdot \begin{pmatrix} x_2 \\ \\ -\alpha x_2^3 - \omega^2 x_1 \end{pmatrix} = -2\alpha x_2^4 \leq 0, \tag{3.5}$$

and is thus a weak, global Lyapunov function for $\mathbf{x}_0$. As in the linear-friction case, the only positively invariant set completely contained in $\mathcal{S} := \{\mathbf{x} \in X : f_{\mathbf{x}}H = 0\} = \{(0, x_2) : x_2 \in \mathbb{R}\}$ is $\{\mathbf{x}_0\}$ its self. By LaSalle's Invariance Principle 3.5, $\mathbf{x}_0$ is globally asymptotically stable.

Note that, in contrast to the previous case, a linear stability analysis fails at identifying the stability behavior of the origin, as the Jacobian

$$A := d_{\mathbf{x}_0} f = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \tag{3.6}$$

being identical to the non-damped case, has purely imaginary eigenvalues $\pm i\omega$.

3.7 Example: Globally, exponentially stable points

Let (X, G) be a differentiable dynamical system induced by the vector field f on the differentiable manifold X. Suppose the topology on X is induced by a metric d. Let $x_0 \in X$ be some point and $H : X \to \mathbb{R}$ a differentiable function satisfying

E.1 $f_x H \leq -\lambda \cdot [H(x) - H(x_0)]$ for all $x \in X$ and some constant $\lambda > 0$.

E.2 $\alpha \cdot [d(x, x_0)]^\beta \leq [H(x) - H(x_0)]$ for all $x \in X$ and some constants $\alpha, \beta > 0$.

Then x_0 is a globally exponentially stable, positively fixed point of the system. More precisely, one has

$$d(G_t(x), x_0) \leq \sqrt[\beta]{\frac{H(x) - H(x_0)}{\alpha}} \cdot e^{-\frac{\lambda}{\beta} t} \tag{3.7}$$

for all $x \in X$ and $t \geq 0$.

Proof: Fix $x \in X$ and define $V(t) := H(G_t(x)) - H(x_0)$ for $t \geq 0$. Then $V : [0, \infty) \to \mathbb{R}$ is differentiable and by axiom (E.1) satisfies $\partial_t V \leq -\lambda V$. This implies $V(t) \leq V(0) \cdot e^{-\lambda t}$, as can be seen by taking the time-derivative of the quotient $V(0)e^{-\lambda t}/V(t)$. By axiom (E.2) this implies

$$\alpha \cdot [d(G_t(x), x_0)]^\beta \leq [H(G_t(x)) - H(x_0)] \leq [H(x) - H(x_0)] \cdot e^{-\lambda t} \tag{3.8}$$

as claimed.

□

3.8 Example: Lyapunov functions for gradient flows

Let (X, G) be a differentiable dynamical system induced by the vector field f on the Riemannian manifold (X, g). Suppose that $f = -\nabla\Phi$ for some differentiable scalar field $\Phi : X \to \mathbb{R}$ (see footnote[1]). Suppose that Φ has an isolated, local minimum at x_0.

Then x_0 is a Lyapunov-stable, positively fixed point of the system. Moreover, if $\nabla_x \Phi \neq 0$ for all $x \neq x_0$ within a neighborhood of x_0, then x_0 is asymptotically stable.

[1] The vector field $\nabla\Phi$ is defined by $d\Phi = g(\nabla f, \cdot)$.

Proof: Let K be a sequentially compact neighborhood of x_0 such that $\Phi(x) > \Phi(x_0)$ for all $x \in K \setminus \{x_0\}$. Now, since $f_x\Phi = -d_x\Phi(\nabla_x\Phi) = -g(\nabla_x\Phi, \nabla_x\Phi) \leq 0$, the restriction $\Phi\big|_K$ satisfies axioms (DWL.1) and (DWL.2) in proposition 3.3. It is thus a weak, local Lyapunov function for the positively fixed point x_0. By theorem 2.4 x_0 is Lyapunov stable.
Now suppose $\nabla_x\Phi \neq 0$ for all $x \neq x_0$ in some neighborhood U of x_0. Then $f_x\Phi < 0$ for all $x \in \overline{K \cap U} \setminus \{x_0\}$, so that axiom (DSL.2) is satisfied and $H\big|_{\overline{K\cap U}}$ is by 3.3 a strong, local Lyapunov function for x_0. By theorem 2.5 x_0 is asymptotically stable.

□

4 Quadratic Lyapunov functions

We shall in this section consider differentiable dynamical systems on $X := \mathbb{R}^n$ induced by a linear vector field $f(\mathbf{x}) = A\mathbf{x}$ ($\mathbf{x} \in X$), with $A \in \mathbb{R}^{n\times n}$ being some constant matrix. In particular, we study the existence of quadratic Lyapunov functions as solutions for the so called *Lyapunov matrix equation*. For more information on the latter and its connection to the stability of linear systems, see [2]. Finally, we apply the theory to the local stability of positively fixed points of non-linear systems.

4.1 Theorem: Lyapunov functions for linear systems

Consider the dynamical system on $X := \mathbb{R}^n$ induced by the linear vector field $f_\mathbf{x} = A\mathbf{x} \in T_\mathbf{x}X$ ($\mathbf{x} \in X$), with $A \in \mathbb{R}^{n\times n}$ being some constant matrix. Let $Q \in \mathbb{R}^{n\times n}$ be some positive definite (non-negative definite) matrix. If there exists a positive definite matrix $M \in \mathbb{R}^{n\times n}$ solving the so called **continuous Lyapunov equation**

$$A^T M + MA + Q = 0, \tag{4.1}$$

then the function $H : X \to \mathbb{R}$ defined by $H(\mathbf{x}) := \mathbf{x}^T M\mathbf{x}$ is a strong (weak), global Lyapunov function for the origin $x_0 := 0$. It satisfies

$$f_\mathbf{x} H = \nabla H\big|_\mathbf{x} \cdot A\mathbf{x} = -\mathbf{x}^T Q\mathbf{x} \tag{4.2}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Proof: Suppose $M \in \mathbb{R}^{n\times n}$ is positive definite and a solution of Eq. (4.1). Then by symmetry of M one finds indeed that

$$\nabla H\big|_\mathbf{x} \cdot A\mathbf{x} = 2\mathbf{x}^T M \cdot A\mathbf{x} = \mathbf{x}^T MA\mathbf{x} + \mathbf{x}^T A^T M^T\mathbf{x} = \mathbf{x}^T(MA + A^T M)\mathbf{x} = -\mathbf{x}^T Q\mathbf{x}. \tag{4.3}$$

Axioms (DSG.1) and (DSG.3) are automatically satisfied by positive definiteness of M and remark 2.1(3). Axiom (DSG.2) is because of Eq. (4.3) verified provided that Q is positive definite, so that H is by proposition 3.2 a strong, global Lyapunov function for $x_0 = 0$. If Q is only non-negative definite, axiom (DWG.2) is verified instead and H is a weak, global Lyapunov function.

□

4.2 Theorem about the continuous Lyapunov equation

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric, positive-definite (non-negative definite) matrix. Let $A \in \mathbb{R}^{n\times n}$ be some matrix with all of it's eigenvalues having a negative real part. Then the integral

$$\int_0^\infty e^{tA^T} Q e^{tA}\, dt =: M \tag{4.4}$$

converges, that is, exists in $\mathbb{R}^{n\times n}$. The so defined matrix M is symmetric and positive-definite (non-negative definite). It is the unique solution of the continuous Lyapunov equation

$$A^T M + MA + Q = 0. \tag{4.5}$$

Proof: The convergence of the integral is given because of the negativity of the real part of all eigenvalues of A. See [3] for more details. The symmetry of M follows from representation (4.4) and the symmetry of Q. Its positive definiteness (non-negative definiteness) follows readily from the fact that $\mathbf{x}^T e^{tA^T} Q e^{tA}\mathbf{x} = (e^{tA}\mathbf{x})^T Q (e^{tA}\mathbf{x})$. The matrix M is indeed a solution of Eq. (4.5), since

$$\begin{aligned} A^T M + MA + Q &= \int_0^\infty \left[A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A^T \right] \, dt + Q \\ &= \int_0^\infty \frac{d}{dt} \left[e^{tA^T} Q e^{tA} \right] \, dt + Q = \lim_{t\to\infty} e^{tA^T} Q e^{tA} - e^{0A^T} Q e^{0A} + Q \overset{(\clubsuit)}{=} 0. \end{aligned} \tag{4.6}$$

Note that in ($\clubsuit$) we used the negativity of the real part of all eigenvalues of A. Since all eigenvalues of A have negative real part, the spectra of A and $-A^T$ are disjoint. By [1], lemma 1, the solution of (4.5) is unique.

□

4.3 Theorem: Quadratic Lyapunov functions for non-linear systems

Let (X, G) be a differentiable dynamical system induced by the differentiable vector field f on some subset $X \subseteq \mathbb{R}^n$. Let $\mathbf{x}_0 \in X^o$ be a positively fixed point of the system and $A := d_{\mathbf{x}_0} f \in \mathbb{R}^{n\times n}$. Let $Q \in \mathbb{R}^{n\times n}$ be some positive definite matrix. If there exists a positive definite matrix $M \in \mathbb{R}^{n\times n}$ solving the continuous Lyapunov equation

$$A^T M + MA + Q = 0, \tag{4.7}$$

then the restriction of the function $H : X \to \mathbb{R},\ H(\mathbf{x}) := (\mathbf{x}-\mathbf{x}_0)^T M (\mathbf{x}-\mathbf{x}_0)$ to some sufficiently small compact neighborhood K of $\mathbf{x}_0$ is a strong, local Lyapunov function for $\mathbf{x}_0$. By theorem 2.5, $\mathbf{x}_0$ is thus asymptotically stable.

Proof: Suppose $M \in \mathbb{R}^{n\times n}$ is positive definite and a solution of Eq. (4.7). Then by symmetry of M one finds that

$$\begin{aligned} \nabla H\big|_{\mathbf{x}} \cdot f_{\mathbf{x}} &= 2(\mathbf{x}-\mathbf{x}_0)^T M \cdot (A(\mathbf{x}-\mathbf{x}_0) + o(\mathbf{x}-\mathbf{x}_0)) \\ &= (\mathbf{x}-\mathbf{x}_0)^T MA(\mathbf{x}-\mathbf{x}_0) + (\mathbf{x}-\mathbf{x}_0)^T A^T M^T (\mathbf{x}-\mathbf{x}_0) + o\big(\|\mathbf{x}-\mathbf{x}_0\|^2 \big) \\ &= (\mathbf{x}-\mathbf{x}_0)^T (MA + A^T M)(\mathbf{x}-\mathbf{x}_0) + o\big(\|\mathbf{x}-\mathbf{x}_0\|^2 \big) \\ &= -(\mathbf{x}-\mathbf{x}_0)^T Q (\mathbf{x}-\mathbf{x}_0) + o\big(\|\mathbf{x}-\mathbf{x}_0\|^2 \big) \end{aligned} \tag{4.8}$$

for every $\mathbf{x} \in X^o$. Eq. (4.8) shows that for small enough deviations $(\mathbf{x}-\mathbf{x}_0) \neq 0$, the directional derivative $\nabla H\big|_{\mathbf{x}} \cdot f_{\mathbf{x}}$ is negative. Thus, axiom (DSL.2) is verified within the interior K^o of some small enough compact neighborhood $K \subseteq$ of $\mathbf{x}_0$.
Axiom (DSL.1) is automatically satisfied by positive definiteness of M. By proposition 3.3, we conclude that $H\big|_K$ is a strong, local Lyapunov function for $\mathbf{x}_0$.

□

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